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Designing a resilient supply chain through a robust adaptive model predictive control policy under perishable goods and uncertain forecast information

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ABSTRACT We deal with the inventory level control problem for Supply Chains (SCs) whose dynamics is affected by two sources of uncertainties: 1) perishable goods with uncertain deterioration rate, 2) an uncertain future customer demand freely varying inside a given bounded set.
The purpose of our contribution is to propose a smooth Replenishment Policy (RP) maximizing the satisfied customer demand and minimizing the inventory level. These requirements should be satisfied despite the above uncertainties and unforeseen customer demand patterns trespassing the "a priori" assumed boundaries. To this purpose we define a Resilient RP (RRP) using a new Robust Adaptive Model Predictive Control (RAMPC) approach. This requires solving a Minimax Constrained Optimization Problem (MCOP). To reduce the complexity of the solving algorithm, we parametrize the predicted replenishment orders in terms of polynomial B-spline basis functions.

INDEX TERMS supply chain management, optimal inventory control, robust adaptive model predictive control, minimax optimization.

I. INTRODUCTION

The numerous applications of MPC to production-inventory control problems are widely recognized and demonstrated by an extensive literature (see e.g. [1], [2] and references therein). The success of MPC is mainly due to : 1) the capability of handling hard constraints imposed on some physical variables, 2) the capability of on-line adapting the actual control action through a receding horizon implementation.
In this context a serious problem is raised by the presence of perishable goods in the inventory system. Due to the widely acknowledged importance of this topic (see e.g. [3]- [6] and references therein), many papers apply MPC techniques to the inventory level of SC's with deteriorating items [7]- [15].

None of the above mentioned papers consider another important source of performance degradation: unpredicted changes of market trend. On the other hand it is of fundamental importance that an SC be endowed with the capacity for resilience, i.e. the ability to promptly react to such events [16]. Extending previous results ([17], [18]), here we also take into account unexpected behaviors of the customer demand.
The main novelties of this contribution with respect to [17]-[18] are:

- endowing the RP with a resilience property to rapidly and effectively recovering from unpredicted and anomalous demand behaviors.
- solving the robust optimization problem considering both sources of uncertainty: deteriorating factor and forecast demand.

A preliminary version of this paper can be found in [19].
On the basis of the previous considerations, the methodological contribution of this paper is the definition of a smooth RRP for the optimal inventory control problem under the following operating conditions: perishable wares with uncertain decay factor, unforeseen customer demand patterns violating some "a priori" assumptions. We adopt an RMPC approach and to allow the SC to quickly react to unpredictable demand patterns we also introduce an adaptation mechanism. The advantage of the resulting RAMPC is the possibility of reconciling opposite control requirements: 1) maximize the satisfied customer demand, 2) avoid an excessive inventory level, 3) produce a smooth RP, 4) fast react to unforeseen patterns of the customer demand.
To greatly reduce the number of calculations involved by the RAMPC, we parametrize the predicted replenishment orders by means of B-splines functions. This choice is due to: 1) B-splines are smooth functions universal approximators of curves with different shape over different intervals, 2) B-

splines admit a parsimonious parametric representation, [20]. Another advantage of this parametrization is the possibility of reformulating the MCOP as a Constrained Robust Least Squares (CRLS) problem that can be efficiently solved using interior-point methods [21]. This also allows us to prove the feasibility of the MCOP and the uniform boundedness of the physical variables (i.e. the inventory level and the control law). These properties are proved without any assumption on the length of the prediction horizon. In this regard, we surprisingly note that two major issues of MPC like stability and feasibility [22] are not explicitly considered in many MPC techniques for SC management. This paper is organized as follows. Mathematical background on B-splines functions and on CRLS is provided in Section II. The dynamic model of the considered uncertain plant is described in Section III. The RAMPC approach is explained in Section IV. The corresponding MCOP is reformulated as a CRLS problem in Section V. Stability and feasibility are proved in Section VI. Numerical simulations are described in Sections VII. Some concluding remarks are drawn in Section VIII.

II. MATHEMATICAL BACKGROUND

A. POLYNOMIAL B-SPLINES FUNCTIONS

An analytic, B-spline function is defined as, [20]:

$$s(t) = \sum_{i=1}^{\ell} c_i B_{i,d}(t), \qquad t \in [\hat{t}_1, \hat{t}_{\ell+d+1}] \subseteq R \tag{1}$$

where the real scalars c_i's are the control points of $s(t)$, the integer d is the degree of the polynomial function $s(t)$, the $(\hat{t}_i)_{i=1}^{\ell+d+1}$ denotes the non decreasing sequence of knot points, the $B_{i,d}(t)$ are the uniformly bounded polynomial B-spline basis functions of degree d. They can be computed through the following recursive formula

$$\begin{aligned} B_{i,d}(t) &= \frac{t-\hat{t}_i}{\hat{t}_{i+d}-\hat{t}_i} B_{i,d-1}(t) \\ &+ \frac{\hat{t}_{i+1+d}-t}{\hat{t}_{i+1+d}-\hat{t}_{i+1}} B_{i+1,d-1}(t), \quad d \geq 1 \end{aligned} \tag{2}$$

with $B_{i,0}(t) = 1$ if $\hat{t}_i \leq t < \hat{t}_{i+1}$, otherwise $B_{i,0}(t) = 0$.
In (2) possible division by zero are resolved by the convention that "anything divided by zero is zero".
An equivalent representation of $s(t)$ in (1) is

$$s(t) = \mathbf{B}_d(t)\mathbf{c}, \qquad t \in [\hat{t}_1, \hat{t}_{\ell+d+1}] \subseteq R \tag{3}$$

where $\mathbf{c} \triangleq [c_1, \cdots, c_\ell]^T$ and $\mathbf{B}_d(t) \triangleq [B_{1,d}(t), \cdots, B_{\ell,d}(t)]$

Convex hull property. All the values of $s(t)$, $\forall t \in [\hat{t}_j, \hat{t}_{j+1}]$, $j > d$, are contained in the convex hull of its $d+1$ control points $c_{j-d}, \cdots, c_j$. △

Smoothness property. If $\hat{t}_i < \hat{t}_{i+1} = \cdots = \hat{t}_{i+m} < \hat{t}_{i+m+1}$, with $1 \leq m \leq d+1$ then the $s(t)$ is continuously derivable up to order $d-m$ at knot $\hat{t}_{i+1}$. This property implies that the spline smoothness can be changed using multiple knot points. It is common choice to set $m = d+1$ multiple knot points for the initial and the last knot points and to evenly distribute the other ones. In this way (1) assumes the first and the final control points as initial and final values. △

Remark 1: From (3) it is apparent that, once the degree d and the knot points $\hat{t}_i$ have been fixed, the scalar B spline function $s(t)$, $t \in [\hat{t}_1, \hat{t}_{\ell+d+1}]$, is completely determined by the corresponding vector $\mathbf{c}$ of ℓ control points. △

The sampled version $s(kT) \triangleq s(k)$, $k \in Z^+$ is obtained by direct uniform sampling (with period T) of the corresponding polynomial B-spline (1).

B. THE CRLS PROBLEM

[21], Consider a set of approximated linear equations $Df \approx b$ where $D \in R^{r\times m}$, with $r > m$, is the design matrix and $b \in R^r$ is the observer vector. Both D and b are affected by bounded uncertainties of unspecified nature: $\|\delta D\| \leq \beta$ and $\|\delta b\| \leq \xi$ (the matrix norm is the spectral norm). The RLS consists in finding the value $\hat{f}$ of $f \in R^m$ solving the following minimax optimization problem

$$\min_f \max_{\|\delta D\|\leq\beta,\ \|\delta b\|\leq\xi} \|(D+\delta D)f - (b+\delta b)\| \tag{4}$$

Using norm properties, it can be shown that

$$\begin{aligned} &\max_{\|\delta D\|\leq\beta,\ \|\delta b\|\leq\xi} \|(D+\delta D)f - (b+\delta b)\| \\ &= \|D\,f - b\| + \beta\|f\| + \xi \end{aligned}$$

Hence $\hat{f}$ can be more easily computed as

$$\min_f \|D\,f - b\| + \beta\|f\| + \xi \tag{5}$$

The CRLS version also requires that f satisfies the following component-wise linear constraints:

$$\underline{f} \leq f \leq \bar{f} \tag{6}$$

Remark 2: Note that the term $\|\delta b\|$ in (4) only appears in (5) through its norm upper bound ξ, which is independent of f. Hence the value of f solving the minimization problem is not altered if ξ is removed from the objective function. In Section V, we show how to solve the MCOP implied by the RAMPC algorithm even in the case of uncertain future customer demand. △

III. THE DYNAMICAL MODEL OF THE SC

Figure 1 shows the structure of the considered SC. A deteriorating item is stored in the retailer stage R, its inventory level is periodically updated at discrete time instants kT, $k \in Z^+$, T is the review period. Explicit dependency on T will be dropped in the following to simplify notation. The following assumptions hold:

- **A1)** the decay factor of the product is $\sigma \in [\sigma^- \ \sigma^+] \subset (0, 1)$;
- **A2)** any replenishment order issued at time k is realized at time $(k+n) \in Z^+$;
- **A3)** the customer demand $w(k)$ is uniformly bounded, and, at any given time instant k, its future trajectory

TABLE 1. Nomenclature

Variables/Parameters/Sets	
$y(k)$	on hand stock level
$w(k)$	customer demand
$v(k)$	fulfilled demand
$u(k)$	replenishment order
n	lead time
$\sigma \in [\sigma^- \sigma^+]$	uncertain decay factor
$\mathcal{J}_k$	functional cost
$\mathcal{U}_k = \{u(j\|k)\}\ j = k, \cdots, k+N-1$	predicted control sequence
$u(j\|k) = B_d(j)c_k$	sample of $\mathcal{U}_k$
$B_d(j)$	B spline basis function
c_k	vector of control points
$e(k+n+l\|k)$	predicted tracking error
u_k^-, u_k^+	bounds on $u(j\|k)$
λ_i, q_i	scalar weights in J_k
P_k	prediction horizon
M	length of P_k
$\mathcal{W}_k$	compact set containing $w(j), j = k, \cdots, k+M$
$w^-(k+\ell), w^+(k+\ell)$	border trajectories of $\mathcal{W}_k$
$\overline{w}(k+\ell)$	central trajectory of $\mathcal{W}_k$
H_k	control horizon
N	length of H_k

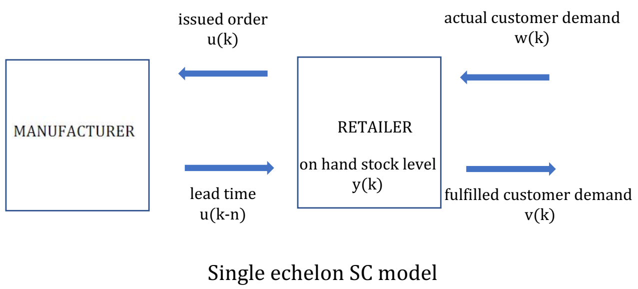
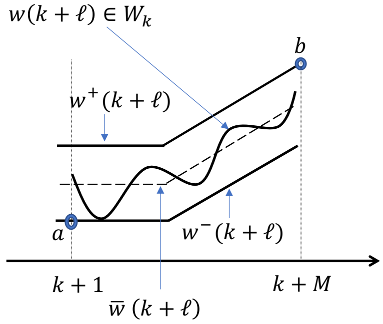
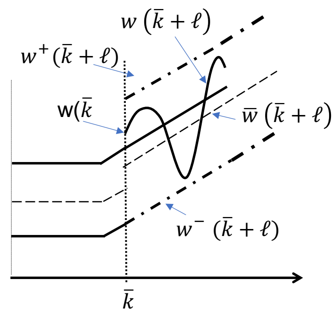


FIGURE 1.



Example of a compact set W_k with time varying border trajectories: $w^-(k+\ell)$ and $w^+(k+\ell)$ ℓ=1,...,M. The dashed trajectory $\overline{w}\,(k+\ell)$ is the middle trajectory of W_k . Points a and b denote w_k^- and w_k^+, respectively. The solid line is the current customer demand.

FIGURE 2.



Example of a customer demand that at time $\bar{k}$ falsifies **A3**. The new set $W_{\bar{k}}$ is defined by two new border trajectories: $w^-\,(\bar{k}+\ell)$ and $w^+\,(\bar{k}+\ell)$ ℓ =1,...,M (dashed-dotted lines).

FIGURE 3.

$w(k+\ell)$, $\ell = 1, \cdots, M$, may freely oscillate inside a known compact set W_k inferiorly and superiorly bounded by two limit trajectories: $w^-(k+\ell)$ and $w^+(k+\ell), \ell = 1, \cdots, M$ respectively. The M-steps time interval $P_k \triangleq [k+1, k+M]$ is called prediction horizon. The minimum value of $w^-(k+\ell)$ and the maximum of $w^-(k+\ell)$ over P_k are denoted by w_k^- and w_k^+ respectively. Figure 2 shows an example of a possible customer demand over a fixed W_k. Figure 3 shows an example of a possible unexpected customer demand violating A3);
- **A4)** inventory replenishment and goods delivery are executed simultaneously at the beginning of each review period. Backorders are not allowed.

By the above modeling assumptions the following uncertain balance equation is directly obtained

$$y(k+1) = \sigma(y(k) + u(k-n) - v(k)),\ y(0) \geq 0, \quad (7)$$

where: $y(k)$ is the on-hand inventory level, $u(k-n)$ is the order issued at time $k-n$ and realized at time k, $v(k)$ is the amount of customer demand that can be satisfied at time k. It is given by

$$v(k) \triangleq \min\{w(k), y_1(k)\} \quad (8)$$

where

$$y_1(k) \triangleq y(k) + u(k-n)$$

is the total quantity of stocked goods available for sale.

IV. PROBLEM SETUP

Given the SC model described in the previous section, the problem we face is to define an RRP endowed with the following Skills: S1) maximize the satisfied customer demand, S2) minimize overstocking, S3) produce a smooth replenishment orders sequence $u(k), k \in Z^+$, S4) quickly adapt to unexpected patterns of the customer demand.

S1 and S2 define the basic requirement for an efficient SC management: satisfy the customer demand at minimum cost, S3 is useful to reduce the costs related to frequent sharp changes in the quantity of ordered goods [23], S4 defines the desired resilience property with respect to unpredicted changes of customer behavior.

The uncertainties on the SC dynamics and the inherent antagonism of the above abilities call for an RAMPC formulation of the above problem. This point is discussed in the next two sections: in section IV-A we define an RP based on an RMPC approach to specifically deal with S1-S3. In section IV-B we define the RRP based on an RAMPC approach endowing the RP with S4.

A. THE RP

The RP is obtained through an RMPC approach. It consists in minimizing, at each k, the maximum value assumed by a suitably defined cost functional for $\sigma \in [\sigma^-, \sigma^+]$ and $w(k+\ell) \in W_k$, $\ell = 1, \cdots, M$. The practical application of the resulting law requires: 1) to repetitively solve at each $k \in Z^+$ an MCOP defined over a future control horizon $H_k \triangleq [k, k+N-1]$, for some $N \leq M$, 2) to only apply the first sample of the computed predicted control sequence $\mathcal{U}_k \triangleq [u(k|k), \cdots, u(k+N-1|k)]$, according to the moving horizon philosophy typical of MPC.

The MCOP is now formally defined on the basis of S1, S2 and S3,

$$\min_{\mathcal{U}_k} \max_{\sigma\in[\sigma^-,\sigma^+], w(\cdot)\in W_k} J_k, \quad k \in Z^+ \tag{9}$$

subject to: $u_k^- \leq u(k+i|k) \leq u_k^+, i = 0, \cdots, N-1$ (10)

$$\begin{aligned} J_k &= \sum_{i=1}^{N} e^T(k+n+i|k) q_i e(k+n+i|k) \\ &+ \sum_{i=1}^{N-1} \lambda_i \Delta u^2(k+i|k) \end{aligned} \tag{11}$$

$$e(k+n+i|k) \triangleq w^+(k+n+i) - y(k+n+i|k) \tag{12}$$

$$\Delta u(k+i|k) \triangleq u(k+i|k) - u(k+i-1|k) \tag{13}$$

$$\begin{aligned} y(k+n+i|k) &\triangleq \sigma^{n+i} y(k) + \sum_{\ell=0}^{n-1} \sigma^{n+i-\ell} u(k+\ell-n) \\ &+ \sum_{\ell=0}^{i-1} \sigma^{i-\ell} u(k+\ell|k) - \sum_{\ell=0}^{n+i-1} \sigma^{n+i-\ell} v(k+\ell|k) \end{aligned} \tag{14}$$

Remark 3:
1) By (11) and (12) we have $M \geq N+n, \forall k \in Z^+$.
2) The signal error (12) has been defined with reference to a time-varying target inventory level given by $w^+(k+n+i)$, $i = 1, \cdots, N$. In the light of S1, S2, and of the bounded uncertainty affecting the customer demand, this is the most appropriate choice: it is useful to maximize the satisfied demand over each $[k+n+1, k+n+N] \subseteq P_k$ and, at the same time, avoids unnecessary larger stock levels.
3) The second term $\sum_{i=1}^{N-1} \lambda_i \Delta u^2(k+i|k)$ of the cost functional penalizes large variations between consecutive values of the control effort. In the light of S3, this is useful to reduce the unavoidable costs related to an RP with sharp discontinuties.
4) The scalar weights q_i, $i = 1, \cdots, N$, and λ_i, $i = 1, \cdots, N-1$, are positive coefficients introduced to progressively decrease the influence of future predictions.
5) The predicted signal $v(k+\ell|k)$ in (14) is computed using (8) under the following assumptions:

- **A5)** $w(k+\ell|k) = \bar{w}(k+\ell)$, $\ell = 1, \cdots, n+i-1$, where $\bar{w}(k+\ell)$ is the central trajectory of W_k, (see figure 2);
- **A6)** $y(k+i) + u(k+i-n) \geq w(k+i)$, $i = 0, \cdots, n+N-1$.

A5) is justified because, in agreement with the minimax approach, $\bar{w}(k+\ell)$, $\ell = 1, \cdots, n+i-1$ is the solution of the following minimax problem:

$$\min_{w(k+\ell|k)} \max_{w(k+\ell)\in W_k} \left(\sum_{\ell=1}^{n+i-1} e_w^T(k+\ell|k) e_w(k+\ell|k) \right)^{1/2} \tag{15}$$

where

$$e_w(k+\ell|k) \triangleq w(k+\ell|k) - w(k+\ell), \ell = 1, \cdots, n+i-1$$

A6) is justified because the control sequence is designed so as to minimize the maximum weighted ℓ_2 norm of a tracking error defined as the difference between the predicted inventory level and the maximum predicted customer demand.
As a consequence of A5), A6) and (8), the predicted term $v(k+\ell|k)$ in (14) can be expressed as

$$v(k+\ell|k) = \bar{v}(k+\ell|k) + \delta v(k+\ell|k) \tag{16}$$

where: $\bar{v}(k|k) = w(k)$, $\delta v(k|k) = 0$, $\bar{v}(k+\ell|k) = \bar{w}(k+\ell)$ and $\delta v(k+\ell|k)$ is the approximation error that, by (15), has the minimum maximum l_2 norm over each P_k.

B. THE RRP

The RRP is obtained through an RAMPC approach. It consists in endowing the RP with a simple adaptive algorithm. The adaptation mechanism is based on the possibility of measuring the disturbance (i.e. $w(k)$) and on the receding horizon nature of MPC. Assume that for some $\bar{k}$, an unexpected demand value $w(\bar{k})$ violating A3) is observed (see figure 3); then a new set $W_{\bar{k}}$ including the measured $w(\bar{k})$ is defined. This is equivalent to redefine A3) coherently with the observed $w(\bar{k})$. As a consequence, also the signal error (12) is defined with respect to a new target inventory level. Owing to the moving horizon implementation of the MPC, the control law is fastly adapted to this unpredicted change with only a minimal computational effort. This procedure is repeated every time the current A3) is violated.
Remark 4: We remark the difficulty of obtaining a similar low cost adaptability using demand forecasting methods based on time series analysis: even using adaptive identification algorithms, the intrinsic inertia of ARMA models slows down the process of adjusting the parameter estimates according to the incoming measures of customer demand.

C. COMPUTING THE HARD CONSTRAINTS ON THE RRP

The predicted control law $u(k+i|k)$ must satisfy (10). These constraints are determined taking into account S1, S2 and the necessity of limiting the amplitude of the interval $[u_k^-, u_k^+]$ to bound the bullwhip effect. Consequently, we want to determine the smallest amplitude interval $[u_k^-, u_k^+]$ that guarantees S1 for all possible customer demands obeying A3). To face the problem raised by the uncertainties on the SC dynamics, we determine u_k^- and u_k^+ making reference to two possible, limit scenarios compatible with the following operating condition of the SC:

- $v(k+i) = w(k+i), i = 0, \cdots, n+N-1$, (according to A6) of Remark 3) ;
- $w(k+i), i = 0, \cdots, n+N-1$, is a constant signal with value $\tilde{w}_k \in [w_k^-, w_k^+]$. The two limit scenarios are $\tilde{w}_k = w_k^-$ and $\tilde{w}_k = w_k^+$.

We now consider the following problem: given a constant $\tilde{w}_k$, we want to determine the corresponding constant control input $\tilde{u}_k$ over each H_k, that guarantees a steady-state value $\tilde{y}_k$ of $y(k)$ satisfying $\tilde{y}_k \geq \tilde{w}_k, \forall \sigma \in [\sigma^-, \sigma^+]$.
This problem can be conveniently solved applying $\mathcal{Z}$-transform methods to (7). To this purpose denote by $W_{u,y}(z) = \frac{\sigma}{z^n(z-\sigma)}$ the transfer function between the $\mathcal{Z}$ transforms of $u(k)$ and $y(k)$, $k \in Z^+$. Analogously let $W_{w,y}(z) = \frac{\sigma}{(z-\sigma)}$ be the transfer function between the $\mathcal{Z}$ transforms of $v(k) = w(k)$ and $y(k)$, $k \in Z^+$. By the final value theorem [24] applied to (7) we have

$$\tilde{y}_k = [W_{u,y}(z)]_{z=1}\tilde{u}_k - [W_{w,y}(z)]_{z=1}\tilde{w}_k, \qquad (17)$$

If σ were exactly known, then, choosing $\tilde{u}_k = \frac{\tilde{w}_k}{\sigma}$, equation (17) would readily imply $\tilde{y}_k = \tilde{w}_k, \forall \tilde{w}_k \in [w_k^-, w_k^+]$. As σ is uncertain, the minimum $\tilde{u}_k$ guaranteeing $\tilde{y}_k \geq \tilde{w}_k, \forall \sigma \in [\sigma^-, \sigma^+]$ is $u_k = \tilde{w}_k/\sigma^-$.
In conclusion, over each H_k we choose u_k^- according to the limit case 1: $\tilde{w}_k = w_k^-$ and u_k^+ according to the limit case 2: $\tilde{w}_k = w_k^+$, obtaining

$$u_k^- \triangleq w_k^-/\sigma^- \leq u(k+i|k) \leq w_k^+/\sigma^- \triangleq u_k^+ \qquad (18)$$
$$k \in Z^+, i = 0, \cdots, N-1$$

The constraints u_k^- and u_k^+ are uniformly bounded as a consequence of the assumed uniform boundedness of the customer demand.

V. FORMULATION OF THE MCOP AS A CRLS PROBLEM

We now show that through a B-splines parametrization of the predicted control sequence $\mathcal{U}_k$, the MCOP (9)-(14) can be reformulated as a CRLS problem. This allows us to drastically reduce the numerical complexity of the algorithm solving the MCOP. Assume to express the sequence $\mathcal{U}_k$ solving the MCOP (9)-(14) as the sampled version of a B-spline function. Hence by (3) we have

$$u(j|k) \triangleq B_d(j)c_k, \quad j = k, \cdots, k+N-1 \qquad (19)$$

The idea is to consider the vector of control points $c_k \triangleq [c_{k,1}, \cdots, c_{k,\ell}]^T$ uniquely defining $u(j|k)$ as the elements of the vector $\hat{f}$ solving (5). This allows us formulating the MCOP as a CRLS problem. Proving this claim requires the following math steps.
As $\sigma \in [\sigma^- \sigma^+]$, then σ^k can be expressed as

$$\sigma^k = (\bar{\sigma} + \delta\sigma)^k = \bar{\sigma}^k + \Delta\sigma_k \qquad (20)$$

where $\bar{\sigma}$ is the nominal value of σ and $\Delta\sigma_k \triangleq (\bar{\sigma}+\delta\sigma)^k - \bar{\sigma}^k$. Hence the free response term of (14) can be expressed as

$$\sigma^{n+i}y(k) = (\bar{\sigma}^{n+i} + \Delta\sigma_{n+i})y(k) \qquad (21)$$

analogously, for the other terms of (14), we have

$$\sum_{\ell=0}^{n-1} \sigma^{n+i-\ell}u(k+\ell-n) = \sum_{\ell=0}^{n-1}(\bar{\sigma}^{n+i-\ell} + \Delta\sigma_{n+i-\ell})u(k+\ell-n) \qquad (22)$$

$$\sum_{\ell=0}^{i-1} \sigma^{i-\ell}u(k+\ell|k) = \sum_{\ell=0}^{i-1}(\bar{\sigma}^{i-\ell} + \Delta\sigma_{i-\ell})B_d(k+\ell)c_k \qquad (23)$$

$$\sum_{\ell=0}^{n+i-1} \sigma^{n+i-\ell}v(k+\ell|k) = \sum_{\ell=0}^{n+i-1} (\bar{\sigma}^{n+i-\ell} + \Delta\sigma_{n+i-\ell})v(k+\ell|k) \qquad (24)$$

Equations (21)-(24) allow us:
1) to separate the terms depending on the control signal (19) from the independent ones;
2) to separate, in either groups of terms, the known quantities from the unknown ones.
Points 1) and 2) allow us to rewrite the tracking error (12) in terms of matrices $D_{k,i}$, $\delta D_{k,i}$ and vectors $b_{k,i}$, $\delta b_{k,i}$and c_k obtaining an expression formally similar to $(D+\delta D)f-(b+\delta b)$ in (4).
First of all, observe that the tracking error (12) can be rewritten as

$$e(k+n+i|k) = (b_{k,i} + \delta b_{k,i}) - (D_{k,i} + \delta D_{k,i})c_k \qquad (25)$$
$$i = 1, \cdots, N$$

where

$$b_{k,i} \triangleq w^+(k+n+i) - \bar{\sigma}^{n+i}y(k) - \sum_{\ell=0}^{n-1} \bar{\sigma}^{n+i-\ell}u(k+\ell-n) + \sum_{\ell=0}^{n+i-1} \bar{\sigma}^{n+i-\ell}\bar{v}(k+\ell|k) \qquad (26)$$

$$\delta b_{k,i} \triangleq -\Delta\sigma_{n+i}y(k) - \sum_{\ell=0}^{n-1} \Delta\sigma_{n+i-\ell}u(k+\ell-n) + \sum_{\ell=0}^{n+i-1} \bar{\sigma}^{n+i-\ell}\delta v(k+\ell|k) + \sum_{\ell=0}^{n+i-1} \Delta\sigma_{n+i-\ell}v(k+\ell|k) \qquad (27)$$

$$D_{k,i} \triangleq \sum_{\ell=0}^{i-1} \bar{\sigma}^{i-\ell} B_d(k+\ell) \tag{28}$$

$$\delta D_{k,i} \triangleq \sum_{\ell=0}^{i-1} \Delta\sigma_{i-\ell} B_d(k+\ell) \tag{29}$$

Equations (26)-(27) have been obtained expressing $v(k+\ell|k)$ as $v(k+\ell|k) = \bar{v}(k+\ell|k) + \delta v(k+\ell|k)$, where $\bar{v}(k+\ell|k)$ and $\delta v(k+\ell|k)$ are defined as in point 5 of Remark 3.
Similarly, the term $\Delta u(k+i|k)$ in (11) can be rewritten as

$$\Delta u(k+i|k) = (b_{u_{k,i}} + \delta b_{u_{k,i}}) - (D_{u_{k,i}} + \delta D_{u_{k,i}})c_k \tag{30}$$
$$i = 1, \cdots, N-1$$

where

$$\begin{aligned} b_{u_{k,i}} &= \delta b_{u_{k,i}} = 0 \\ D_{u_{k,i}} &= -\left(B_d(k+i) - B_d(k+i-1)\right) \\ \delta D_{u_{k,i}} &= 0 \end{aligned}$$

Define the following vectors and matrices

$$\underline{e}_k = \begin{bmatrix} q_1^{1/2} e(k+n+1|k) \\ \vdots \\ q_N^{1/2} e(k+n+N|k) \\ \lambda_1^{1/2} \Delta u(k+1|k) \\ \vdots \\ \lambda_{N-1}^{1/2} \Delta u(k+N-1|k) \end{bmatrix}, \underline{D}_k = \begin{bmatrix} q_1^{1/2} D_{k,1} \\ \vdots \\ q_N^{1/2} D_{k,N} \\ \lambda_1^{1/2} D_{u_{k,1}} \\ \vdots \\ \lambda_{N-1}^{1/2} D_{u_{k,N-1}} \end{bmatrix}$$

$$\underline{b}_k = \begin{bmatrix} q_1^{1/2} b_{k,1} \\ \vdots \\ q_N^{1/2} b_{k,N} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{\delta b}_k = \begin{bmatrix} q_1^{1/2} \delta b_{k,1} \\ \vdots \\ q_N^{1/2} \delta b_{k,N} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{31}$$

$$\underline{\delta D}_k = \begin{bmatrix} q_i^{1/2} \delta D_{k,1} \\ \vdots \\ q_N^{1/2} \delta D_{k,N} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{32}$$

Exploiting the above defined vectors and matrices, allows us to express the $2N-1$ equations (25) and (30) in a compact matrix form and to reformulate the MCOP (9)-(14) as:

$$\min_{c_k} \max_{\|\underline{\delta D}_k\| \le \beta_k \, \|\underline{\delta b}_k\| \le \xi_k} J_k \tag{33}$$

where

$$J_k = \|(\underline{b}_k + \underline{\delta b}_k) - (\underline{D}_k + \underline{\delta D}_k)c_k\|^2 \tag{34}$$
$$\text{subject to } \; u_k^- \le c_{k,j} \le u_k^+, j = 1, \cdots, \ell \tag{35}$$

Constraints (35) are a direct consequence of (19) and of the convex hull property of B splines.
It is seen that solution of (33), (34) subject to (35) is the same of a problem of the kind (4) subject to (6). Therefore the solution of the MCOP (9)-(14) is given by the vector c_k solving the following CRLS problem:

$$\min_{c_k} \|\underline{b}_k - \underline{D}_k c_k\| + \beta_k \|c_k\| + \xi_k, \quad k \in Z^+, \tag{36}$$

subject to (35).
The above calculations show that the solution of the RAMPC can be lead back to that of a CRLS problem for which very numerical efficient interior point methods exist.

In conclusion, each sample $u(k)$ of the RRP results from the receding horizon implementation of $\mathcal{U}_k$ and from a time-varying target inventory level defined as specified in Section IV-B.

Remark 5: Recalling the minimax problem (15), and taking into account the way the vector $\underline{\delta b}_k$ is defined by (27) and (31), it is seen that, choosing $v(k+\ell|k)$ as in (16), the corresponding upper bound ξ_k of $\|\underline{\delta b}_k\|$ takes its minimum value. As a consequence also the term ξ_k that in (36) is independent of c_k is minimized. This implies that we actually solve the MCOP problem (9),(14).

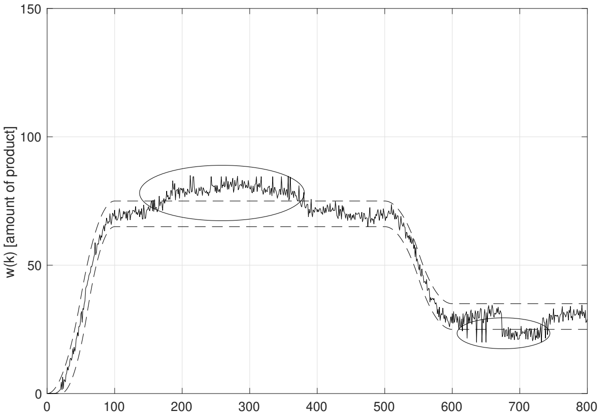
VI. THE STABILITY AND FEASIBILITY ISSUE

The following theorem holds.
Theorem The MCOP is feasible and the physical variables $u(k)$ and $y(k)$ are uniformly bounded.
Proof The feasibility of the MCOP is a consequence of modeling $u(j|k)$ as in (19): the vector c_k solves the equivalent CRLS problem (33), and, at the same time, satisfies (35). Hence also (10) is satisfied by the convex hull property of B-splines.
The uniform boundedness of $u(k)$ follows from:
- 1) It is obtained by the receding horizon implementation of $u(j|k)$;
- 2) $u(j|k)$ is uniformly bounded as a direct consequence of (19) , (35), the uniform boundedness of u_k^- and u_k^+ and the convex hull property of B-splines.
The uniform boundedness of $y(k)$ follow as a direct consequence of the internal asymptotic stability of the SC model ($\sigma < 1$).

VII. SIMULATION RESULTS AND DISCUSSION

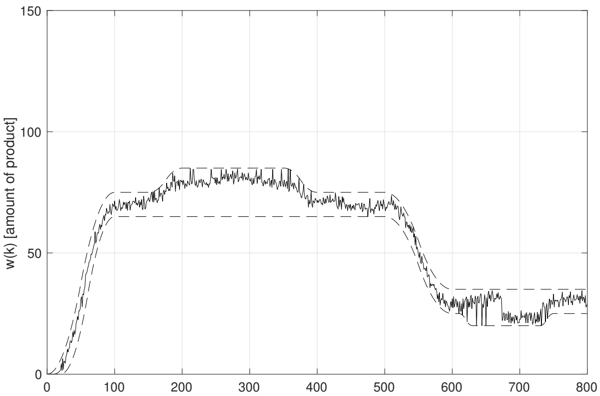
The simulations reported in this section have been implemented using Matlab R2018b.

The considered SC model is defined by a time delay $n = 5$, and an uncertain decay factor $\sigma \in [\sigma^-, \sigma^+] = [0.86, 0.9]$. At each k, the future values of $w(k)$, vary inside a compact set W_k, with length $M = 17$, like the example shown in Figure 2. The whole trajectory of the actual $w(k)$ considered in this simulation is reported in Figure 4. It shows two unforeseen patterns over the intervals $[k_1 \; k_2] = [170 \; 380]$ and $[k_3 \; k_4] = [630 \; 750]$ respectively. Figure 5 shows the new



The actual customer demand w(k) (solid line) with unexpected behaviors (oval curve)

FIGURE 4.



The actual customer demand w(k) (solid line),
the new upper $w^+(k)$ and the new lower $w^-(k)$ boundaries (dashed lines)

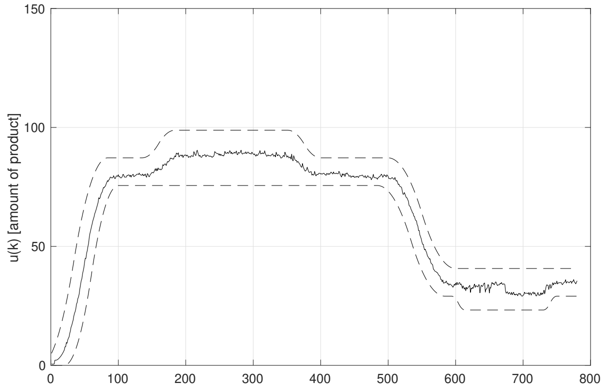
FIGURE 5.

set W_k resulting from a consecutive contiguous overlapping of all the W_k's enclosing the whole actual $w(k)$ (see Section IV-B).

The control algorithm generating the RRP is defined by the following parameters:
- degree of splines: $d = 3$,
- number of control points over each control horizon H_k: $\ell = 6$,
- length of the control horizon H_k = $N = M - n = 12$, (see Remark 3),
- $y(0) = 0$,
- coefficient weighting the i-th element $e(k + n + i|k)$: $q_i = e^{-0.1\ (i-1)}$,
- coefficient weighting the i-th element $\Delta u(k + i|k)$: $\lambda_i = e^{-1\ (i-1)}$.

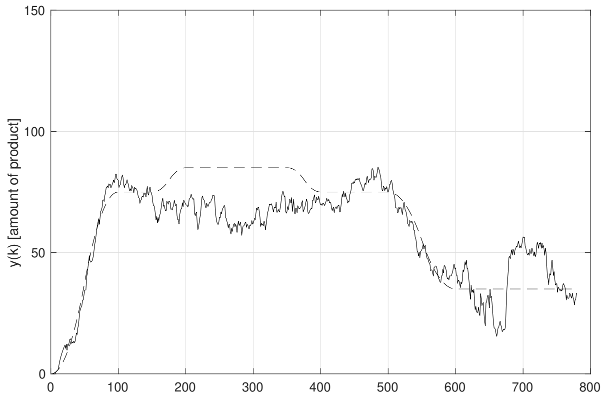
The model equation (7) has been implemented assuming $\sigma = 0.885$. The obtained RRP and its constraints are shown in Figure 6. The solid line reported in 7 is the actual inventory level $y(k)$, the dashed line is the corresponding target trajectory $\tilde{y}(k) = w^+(k)$. The satisfied customer demand is reported in Figure 8.

This figure evidences the resilience property of the proposed RP: the adaptation mechanism of Section IV-B allows a full satisfaction of the customer demand though the presence of two unpredicted patterns.



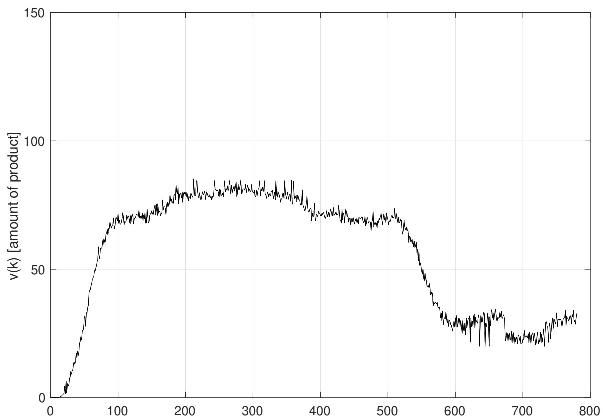
(RAMPC) The sequence of generated orders u(k) (solid line)
and the relative constraints trajectories: u_k^- and u_k^+ (dashed lines).

FIGURE 6.



(RAMPC) The current inventory level y(k) ((solid line) and
the desired time varying inventory level $w^+(k)$ (dashed line)

FIGURE 7.



(RAMPC) The satisfied demand v(k) coinciding with the actual demand w(k)

FIGURE 8.

A. COMPARISON WITH THE ORDER-UP-TO (OUT) RP

We compared the RRP with the RP generated by

$$u(k) = (\tilde{y} - \sigma^{n+1}y(k) - \sum_{\ell=2}^{n+1} \sigma^{\ell} u(k-\ell+1))/\sigma \qquad (37)$$

where $\tilde{y}$ is the constant target inventory level. Equation (37) is a modified version of the classical OUT RP [25] that has been adapted to the uncertain, retarded plant equation (7). The values $\tilde{y} = 300$ and $\sigma = 0.88$ have been chosen. To increase the amount of satisfied demand, the reference level has been chosen significantly larger than the maximum customer demand over the whole simulation period ($\max_k w^+(k) = 85$). The model equation (7) has been implemented assuming: $y(0) = 0$ and $\sigma = 0.885$. The generated RS $u(k)$ and the corresponding inventory level $y(k)$ are reported in Figures 9 and 10 respectively. Figure 11 highlights that the OUT RP is not able to fulfill the customer demand in the correspondence of the unpredicted pattern over the interval $[170\ 380]$. Note that this occurs though the considerably larger value of the desired inventory level (300 for the OUT RP and $\max_k w^+(k) = 85$ for our method).

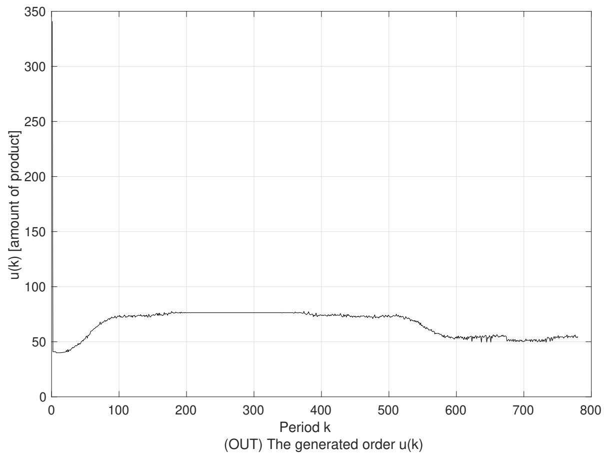


FIGURE 9.

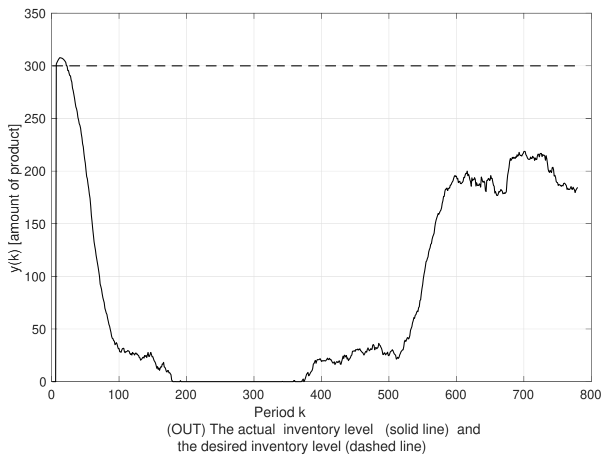


FIGURE 10.

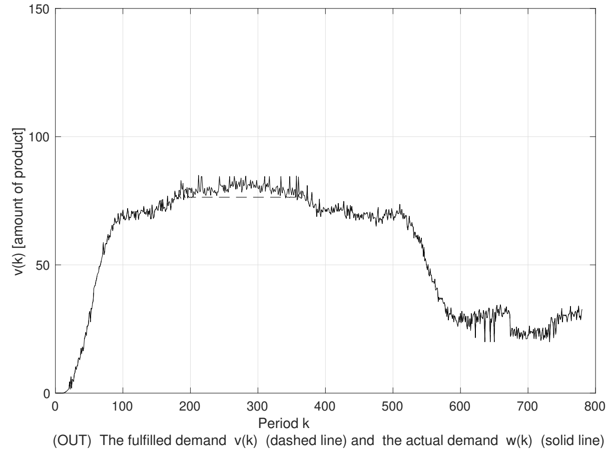


FIGURE 11.

Figure 12 shows that the RRP provides a smoother control signal with respect to the OUT RP. This is due to : 1) introducing the term $\sum_{i=1}^{N-1} \lambda_i \Delta u^2(k+i|k)$ in the cost functional (11), 2) using polynomial B-splines to express $u(j|k)$.

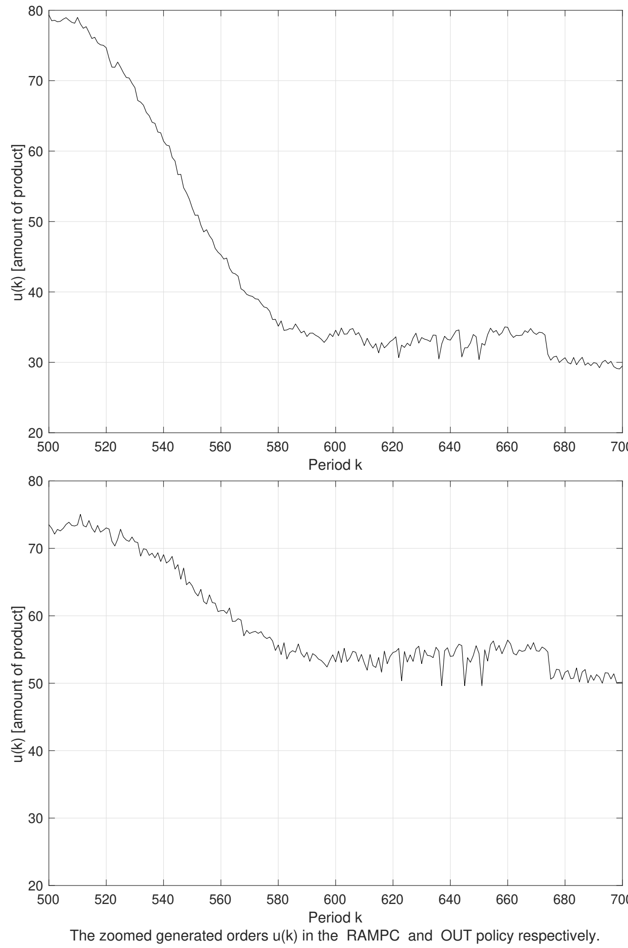


FIGURE 12.

The enhanced degree of smoothness deriving from the RRP decreases the order quantity changes with respect to the OUT

RP.

VIII. CONCLUSIONS

We considered an SC inventory management problem characterized by the following elements of complexity: 1) perishable stocked goods with uncertain decay factor, 2) an uncertain future customer demand that may exhibit unexpected behaviors. Using a RAMPC approach we proposed a solution consisting of an RRP balancing the opposite needs of low cost inventory holding and low percentage of lost sales. The RRP is also able to quickly recover from unexpected changes of customer demand. The numerical simulations confirmed the effectiveness of the approach.

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